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General Equations of Motion for Test Particles in Space-Time with Torsion

Chen Fang Pei[1]

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The momentum and spin equations of motion for test particles possessing different spins in space-time with torsion are derived from the most general functional form of $\mathscr{L}_M$. The same kinds of equations in general relativity and in Kibble's gauge theory of gravitation are special cases of our equations.

1. THE MOST GENERAL FUNCTIONAL FORM OF $\mathscr{L}_M$

It is well known that the Lagrangian density $\mathscr{L}$ in the theory of gravitation can be split into two parts:

$$\mathscr{L} = \mathscr{L}_G + \mathscr{L}_M$$

$\mathscr{L}_G$ is the gravitational part, which contains only gravitational fields; $\mathscr{L}_M$ is the part relating to matter and interaction, which contains matter fields and gravitational fields due to the interaction between matter fields and gravitational fields. In the general case, the gravitational fields are represented by the vierbein field $h^i_\mu(x)$ and the frame connection field $\Gamma^{ij}_\mu(x)$. The metric field $g_{\mu\nu}(x)$ can be expressed by

$$g_{\mu\nu}(x) = h^i_\mu(x) h^j_\nu(x) \eta_{ij}(x) \tag{1}$$

and the holonomic connection field $\Gamma^\mu_{\nu\lambda}(x)$ is related to $h^i_\mu(x)$ and $\Gamma^{ij}_\mu(x)$ by (Chen and Deng, 1988)

$$\Gamma^\mu_{\nu\lambda}(x) = h_i{}^\mu(x)[h^i_{\nu,\lambda}(x) + \Gamma^i_{j\lambda}(x) h^j_\nu(x)] \tag{2}$$

In the space-time with torsion the most general functional form of $\mathscr{L}_M$ may be denoted by

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), \partial_\mu\psi(x), h^i_\mu(x), h^i_{\mu,\nu}(x), \Gamma^{ij}_\mu(x), \Gamma^{ij}_{\mu,\nu}(x), \eta_{ij}(x), \gamma^i(x)] \tag{3}$$

[1]Physics Department, Dalian University of Technology, China.

where $\psi(x)$, $\eta_{ij}(x)$, and $\gamma^i(x)$ denote matter fields, the local Minkowski metric, and local Dirac matrices, respectively. The matter fields may be tetrad tensors (include scalars and vectors) or tetrad spinors, but I suppose they are coordinate scalars; if they are only tetrad tensors, then γ^i matrices will not appear in $\mathscr{L}_M$.

The functional forms of $\mathscr{L}_M$ given by

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), \Gamma^{ij}_\mu(x), \eta_{ij}(x), \gamma^i(x)] \tag{4}$$

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), h^i_{\mu,\nu}(x), \eta_{ij}(x), \gamma^i(x)] \tag{5}$$

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), \eta_{ij}(x)] \tag{5$'$}$$

are all special cases of (3); (4) represents $\mathscr{L}_M$ in Kibble's gauge theory of gravitation (KGTG); (5) and (5′) represent $\mathscr{L}_M$ of nonscalar fields and scalar fields, respectively, in general relativity (GR). In KGTG and in GR, $\mathscr{L}_M$ can also be expressed by

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), h^\mu_k(x)\psi_{|\mu}(x), \eta_{ij}(x), \gamma^i(x)] \tag{6}$$

where

$$\psi_{|\mu}(x) = \psi_{,\mu}(x) + \tfrac{1}{2}\Gamma^{ij}_\mu(x)\sigma_{ij}\psi(x) \tag{7}$$

for KGTG (Kibble, 1961), σ_{ij} are the generators of the Lorentz group; and

$$\psi_{|\mu}(x) = \psi_{,\mu}(x) + \tfrac{1}{2}h^\lambda_i(x)h_{j\lambda,\mu}(x)\sigma^{ij}\psi(x) \tag{8}$$

for the general case of GR (Weinberg, 1972) after choosing appropriate coordinates, $\sigma^{ij} = \eta^{ik}\eta^{jl}\sigma_{kl}$. For the scalar field in GR, $\sigma^{ij} = 0$, (8) reduces to

$$\psi_{|\mu}(x) = \psi_{,\mu}(x) \tag{8$'$}$$

We can write (Kibble, 1961)

$$h^\mu_k(x)\psi_{|\mu}(x) = \delta^\mu_k\,\psi_{,\mu}(x) + \tfrac{1}{2}\Gamma^{ij}_k(x)\sigma_{ij}\psi(x) - [\delta^\mu_k - h^\mu_k(x)]\,\partial_\mu\psi(x) \tag{9}$$

for KGTG and write

$$\begin{aligned} h^\mu_k(x)\psi_{|\mu}(x) = {} & \delta^\mu_k\psi_{,\mu}(x) + \tfrac{1}{2}h^\mu_k(x)h^\lambda_i(x)h_{j\lambda,\mu}(x)\sigma^{ij}\psi(x) \\ & - [\delta^\mu_k - h^\mu_k(x)]\,\partial_\mu\psi(x) \end{aligned} \tag{10}$$

for GR; these relations tell us that the couplings between matter fields and gravitational fields in KGTG are represented by $\Gamma^{ij}_k(x)\sigma_{ij}\psi(x)$ and

$$[\delta^\mu_k - h^\mu_k(x)]\,\partial_\mu\psi(x),$$

but those in GR are represented by

$$h_k^\mu(x)h_i^\lambda(x)h_{j\lambda,\mu}(x)\sigma^{ij}\psi(x)$$

and $[\delta_k^\mu - h_k^\mu(x)]\,\partial_\mu\psi(x)$.

We must point out that as (3) is the most general form of $\mathscr{L}_M$, it cannot be expressed by (7); besides $[\delta_k^\mu - h_k^\mu(x)]\,\partial_\mu\psi(x)$, $\Gamma_k^{ij}(x)\sigma_{ij}\psi(x)$, or $h_k^\mu(x)h_i^\lambda(x)h_{j\lambda,\mu}(x)\sigma^{ij}\psi(x)$, (3) could include types of coupling terms such as $bR(x)|\psi(x)|^2$ or $cT^\mu_{\nu\lambda}(x)T^{\nu\lambda}_\mu(x)|\psi(x)|^2$, etc.; $R(x)$ and $T^\mu_{\nu\lambda}(x)$ are the scalar curvature and the torsion tensor of space-time, respectively, and b and c are coupling constants. The Lagrangian density of scalar field given by Birrel and Davies (1982)

$$\mathscr{L}_M(x) = \tfrac{1}{2}[-g(x)]^{1/2}\{g^{\mu\nu}(x)\phi_{,\mu}(x)\phi_{,\nu}(x) - [m^2 + \xi R(x)]\phi^2(x)\}$$

is an example of (3). In this paper I do not intend to study the concrete forms of coupling between matter fields and gravitational fields, but am only interested in the general equations of motion deduced from (3) for test particles.

Another special case of (3) is

$$\mathscr{L}_M(x) = \mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), h^i_{\mu,\nu}(x), \Gamma^{ij}_\mu(x), \eta_{ij}(x), \gamma^i(x)] \quad (11)$$

which is discussed elsewhere (Chen and Jiang, 1989).

2. SYMMETRY AND IDENTITIES

Suppose that the action integral

$$I = \int \mathscr{L}_M(x)\, d^4x$$

is invariant under both the local Lorentz transformation of the tetrad frame and the general coordinate transformation simultaneously. Let the infinitesimal variations of these transformations be

$$e_i(x) \to e_i'(x') = e_i(x) - \varepsilon^{mn}(x)\delta^j_m\eta_{ni}e_j(x) \quad (12)$$

and

$$\chi^\mu \to \chi'^\mu = \chi^\mu + \xi^\mu(x) \quad (13)$$

where $\varepsilon^{mn}(x)$ $[= -\varepsilon^{nm}(x)]$ and $\xi^\mu(x)$ are local parameters which specify the local Lorentz transformation of the tetrad frame $\{e_i\}$ and the general coordinate transformation, respectively. The induced variations of $\psi(x)$,

$\psi_{,\mu}(x)$, $h^i_\mu(x)$, $h^i_{\mu,\lambda}(x)$, $\Gamma^{ij}_\mu(x)$, $\Gamma^{ij}_{\mu,\lambda}(x)$, $\eta_{ij}(x)$, $\gamma^i(x)$, and $\mathscr{L}_M(x)$ are

$$\delta\psi(x)=\tfrac{1}{2}\varepsilon^{mn}(x)\sigma_{mn}\psi(x) \tag{14}$$

$$\delta\partial_\mu\psi(x)=\tfrac{1}{2}\varepsilon^{mn}_{,\mu}(x)\sigma_{mn}\psi(x)+\tfrac{1}{2}\varepsilon^{mn}(x)\sigma_{mn}\psi_{,\mu}(x)-\xi^\nu_{,\mu}(x)\psi_{,\nu}(x) \tag{15}$$

$$\delta h^i_\mu(x)=\varepsilon^{mn}(x)\delta^i_{[m}\eta_{n]j}h^j_\mu(x)-\xi^\nu_{,\mu}(x)h^i_\nu(x) \tag{16}$$

$$\begin{aligned}\delta h^i_{\mu,\lambda}(x)&=\varepsilon^{mn}_{,\lambda}(x)\delta^i_{[m}\eta_{n]j}h^j_\mu(x)+\varepsilon^{mn}(x)\delta^i_{[m}\eta_{n]j}h^j_{\mu,\lambda}(x)\\&\quad-\xi^\nu_{,\mu\lambda}(x)h^i_\nu(x)-\xi^\nu_{,\mu}(x)h^i_{\nu,\lambda}(x)-\xi^\nu_{,\lambda}(x)h^i_{\mu,\nu}(x)\end{aligned} \tag{17}$$

$$\begin{aligned}\delta\Gamma^{ij}_\mu(x)&=\varepsilon^{mn}(x)\delta^i_{[m}\eta_{n]k}\Gamma^{kj}_\mu(x)+\varepsilon^{mn}(x)\delta^j_{[m}\eta_{n]k}\Gamma^{ik}_\mu(x)\\&\quad-\xi^\nu_{,\mu}(x)\Gamma^{ij}_\nu(x)-\varepsilon^{mn}_{,\mu}(x)\delta^i_{[m}\delta^j_{n]}\end{aligned} \tag{18}$$

$$\begin{aligned}\delta\Gamma^{ij}_{\mu,\lambda}(x)&=\varepsilon^{mn}_{,\lambda}(x)\{\delta^i_{[m}\eta_{n]k}\Gamma^{kj}_\mu(x)+\delta^j_{[m}\eta_{n]k}\Gamma^{ik}_\mu(x)\}\\&\quad+\varepsilon^{mn}(x)\{\delta^i_{[m}\eta_{n]k}\Gamma^{kj}_{\mu,\lambda}(x)+\delta^j_{[m}\eta_{n]k}\Gamma^{ik}_{\mu,\lambda}(x)\}\\&\quad-\xi^\nu_{,\mu\lambda}(x)\Gamma^{ij}_\nu(x)-\xi^\nu_{,\mu}(x)\Gamma^{ij}_{\nu,\lambda}(x)-\varepsilon^{mn}_{,\mu\lambda}(x)\\&\quad\times\delta^i_{[m}\delta^j_{n]}-\xi^\nu_{,\lambda}(x)\Gamma^{ij}_{\mu,\nu}(x)\end{aligned} \tag{19}$$

$$\delta\eta_{ij}(x)=0$$

$$\delta\gamma^i(x)=0$$

$$\begin{aligned}\delta\mathscr{L}_M(x)&=\frac{\partial\mathscr{L}_M}{\partial\psi(x)}\,\delta\psi(x)+\frac{\partial\mathscr{L}_M}{\partial\psi_{,\mu}(x)}\,\delta\psi_{,\mu}(x)+\frac{\partial\mathscr{L}_M}{\partial h^i_\mu(x)}\,\delta h^i_\mu(x)\\&\quad+\frac{\partial\mathscr{L}_M}{\partial h^i_{\mu,\lambda}(x)}\,\delta h^i_{\mu,\lambda}(x)+\frac{\partial\mathscr{L}_M}{\partial\Gamma^{ij}_\mu(x)}\,\delta\Gamma^{ij}_\mu(x)\\&\quad+\frac{\partial\mathscr{L}_M}{\partial\Gamma^{ij}_{\mu,\lambda}(x)}\,\delta\Gamma^{ij}_{\mu,\lambda}(x)\end{aligned} \tag{20}$$

The square brackets $[\cdot]$ in (16)-(19) denote the operation of antisymmetrization. The Lorentz group generators σ_{mn} appearing in (14) and (15) are

$$\sigma_{mn}=\tfrac{1}{2}\gamma_{[m}\gamma_{n]} \qquad \text{for spinor fields}$$

$$\sigma_{mn}=0 \qquad \text{for scalar fields}$$

$$[\sigma_{mn}]^i_j=\delta^i_m\eta_{nj}-\delta^i_n\eta_{mj} \qquad \text{for vector fields, etc.}$$

equations (16) and (18) can be derived from the following relations:

$$V^i(x)=h^i_\mu(x)V^\mu(x)$$

$$V^i_{|\mu}(x)=V^i_{,\mu}(x)+\Gamma^i_{j\mu}(x)V^j(x)$$

$$\Gamma^{ij}_\mu(x)=\eta^{jk}\Gamma^i_{k\mu}(x)$$

equations (15), (17), and (19) can be derived from the relation

$$\delta(A_{,\mu}(x)) = (\delta A(x))_{,\mu} - \xi^{\nu}_{,\mu}(x) A_{,\nu}(x)$$

where $V^i(x)$ [or $V^\mu(x)$] is a vector field and $A(x)$ is a geometric object field.

The sufficient condition of the action integral invariance is the relation (Corson, 1953)

$$\delta\mathscr{L}_M + \xi^{\nu}_{,\nu}\mathscr{L}_M = 0 \tag{21}$$

Putting (14)-(20) into (21), using the independent arbitrariness of $\varepsilon^{mn}(x)$, $\varepsilon^{mn}_{,\mu}(x)$, $\varepsilon^{mn}_{,\mu\lambda}(x)$, $\xi^{\nu}_{,\mu}(x)$, and $\xi^{\nu}_{,\mu\lambda}(x)$, and utilizing the field equation

$$\frac{\delta\mathscr{L}_M}{\delta\psi} = \frac{\partial\mathscr{L}_M}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}_M}{\partial\psi_{,\mu}}\right) = 0 \tag{22}$$

and the definitions

$$\mathfrak{T}^{\mu}_{i} := \frac{\delta\mathscr{L}_M}{\delta h^i_\mu} = \frac{\partial\mathscr{L}_M}{\partial h^i_\mu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial\mathscr{L}_M}{\partial h^i_{\mu,\lambda}}\right) \tag{23}$$

$$\mathfrak{C}^{\mu}_{mn} := -2\frac{\delta\mathscr{L}_M}{\delta\Gamma^{mn}_{\mu}} = -2\left\{\frac{\partial\mathscr{L}_M}{\partial\Gamma^{mn}_{\mu}} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial\mathscr{L}_M}{\partial\Gamma^{mn}_{\mu,\lambda}}\right)\right\} \tag{24}$$

$$\mathfrak{S}^{\mu}_{mn} := -\frac{\partial\mathscr{L}_M}{\partial\psi_{,\mu}} S_{mn}\psi \tag{25}$$

$$\mathscr{B}^{\mu}_{mn} := -\left(\frac{\partial\mathscr{L}_M}{\partial h^m_{\nu,\mu}} h_{n\nu} - \frac{\partial\mathscr{L}_M}{\partial h^n_{\nu,\mu}} h_{m\nu}\right) \tag{26}$$

$$\mathscr{D}^{\mu}_{mn} := -2\left(\frac{\partial\mathscr{L}_M}{\partial\Gamma^{am}_{\nu,\mu}}\Gamma^{a}_{n\nu} - \frac{\partial\mathscr{L}_M}{\partial\Gamma^{an}_{\nu,\mu}}\Gamma^{a}_{m\nu}\right) + 2\frac{\partial}{\partial x^\lambda}\left(\frac{\partial\mathscr{L}_M}{\partial\Gamma^{mn}_{\mu,\lambda}}\right) \tag{27}$$

and the antisymmetric property of Γ^{ij}_{μ} (Yasskin, 1979),

$$\Gamma^{ij}_{\mu} = -\Gamma^{ji}_{\mu} \tag{28}$$

we obtain the following identities:

$$\frac{\partial}{\partial x^\mu}\mathfrak{C}^{\mu}_{mn} = 2\mathfrak{T}_{[mn]} + \Gamma^{1}_{m\mu}\mathfrak{C}^{\mu}_{ln} + \Gamma^{1}_{n\mu}\mathfrak{C}^{\mu}_{ml} \tag{29}$$

$$\mathfrak{C}^{\mu}_{mn} = \mathfrak{C}^{\mu}_{mn} + \mathfrak{B}^{\mu}_{mn} + \mathfrak{D}^{\mu}_{mn} \tag{30}$$

$$\frac{\partial\mathscr{L}_M}{\partial\Gamma^{mn}_{\mu,\lambda}} = -\frac{\partial\mathscr{L}_M}{\partial\Gamma^{mn}_{\lambda,\mu}} \tag{31}$$

$$\begin{aligned}\mathfrak{T}^{\mu}_{\nu} = {}& \delta^{\mu}_{\nu}\mathscr{L}_M - \frac{\partial\mathscr{L}_M}{\partial\psi_{,\mu}}\psi_{,\nu} - \frac{\partial\mathscr{L}_M}{\partial h^i_{\alpha,\mu}} h^i_{\alpha,\nu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial\mathscr{L}_M}{\partial h^i_{\mu,\lambda}} h^i_\nu\right) \\ & + \tfrac{1}{2}\mathfrak{C}^{\mu}_{ij}\Gamma^{ij}_{\nu} - \frac{\partial\mathscr{L}_M}{\partial\Gamma^{ij}_{\alpha,\mu}}\Gamma^{ij}_{\alpha,\nu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial\mathscr{L}_M}{\partial\Gamma^{ij}_{\mu,\lambda}}\Gamma^{ij}_{\nu}\right)\end{aligned} \tag{32}$$

and

$$\frac{\partial \mathscr{L}_M}{\partial h^i_{\mu,\lambda}} = -\frac{\partial \mathscr{L}_M}{\partial h^i_{\lambda,\mu}} \tag{33}$$

where

$$\mathfrak{T}^\mu_\nu = h^i_\nu \mathfrak{T}^\mu_i, \qquad \mathfrak{T}_{mn} = \eta_{in} h^i_\mu \mathfrak{T}^\mu_m$$

All of $\mathfrak{T}^\mu_i$, $\mathfrak{T}^\mu_\nu$ (or $\mathfrak{T}^{\nu\mu} = g^{\nu\sigma}\mathfrak{T}^\mu_\sigma$), and $\mathfrak{T}_{mn}$ are energy-momentum tensor densities of the matter field, but in different indices; *i*, *m*, *n*, and other Latin letters are frame indices; μ, ν, σ, and other Greek letters are coordinate indices. $\mathfrak{T}^{\nu\mu}$ or $\mathfrak{T}_{mn}$ may be asymmetric in the general case; therefore, in this paper, I consider the possible existence of an asymmetric energy-momentum tensor density and consider its symmetric circumstance as a special case. Equation (25) is the ordinary definition of the spin density of the matter field in field theory and $\mathfrak{S}^\mu_{mn}$ is called the ordinary spin density. The $\mathfrak{C}^\mu_{mn}$ is the generalized spin density and $\mathfrak{B}^\mu_{mn}$ and $\mathfrak{D}^\mu_{mn}$ are additive spin densities caused by the vierbein field and the frame connection field, respectively. For scalar fields $\mathfrak{S}^\mu_{mn} = 0$, but $\mathfrak{C}^\mu_{mn} \neq 0$ in general in the present theory.

Since $\mathscr{L}_M$ does not depend on x explicitly, we have

$$\begin{aligned}\frac{\partial}{\partial x^\nu} \mathscr{L}_M(x) &= \frac{\partial \mathscr{L}_M}{\partial \psi} \psi_{,\nu} + \frac{\partial \mathscr{L}_M}{\partial \psi_{,\mu}} \psi_{,\mu\nu} + \frac{\partial \mathscr{L}_M}{\partial h^i_\mu} h^i_{\mu,\nu} \\ &\quad + \frac{\partial \mathscr{L}_M}{\partial h^i_{\mu,\lambda}} h^i_{\mu,\lambda\nu} + \frac{\partial \mathscr{L}_M}{\partial \Gamma^{ij}_\mu} \Gamma^{ij}_{\mu,\nu} + \frac{\partial \mathscr{L}_M}{\partial \Gamma^{ij}_{\mu,\lambda}} \Gamma^{ij}_{\mu,\lambda\nu}\end{aligned} \tag{34}$$

Using (22)-(24), equation (34) can be transformed into

$$\frac{\partial}{\partial x^\mu}\left(\delta^\mu_\nu \mathscr{L}_M - \frac{\partial \mathscr{L}_M}{\partial \psi_{,\mu}} \psi_{,\nu} - \frac{\partial \mathscr{L}_M}{\partial h^i_{\alpha,\mu}} h^i_{\alpha,\nu} - \frac{\partial \mathscr{L}_M}{\partial \Gamma^{ij}_{\alpha,\mu}} \Gamma^{ij}_{\alpha,\nu}\right) = \mathfrak{T}^\mu_i h^i_{\mu,\nu} - \tfrac{1}{2}\mathfrak{C}^\mu_{ij} \Gamma^{ij}_{\mu,\nu} \tag{35}$$

From (35) and (32) and utilizing (31), (33), (29), (28), and (2), we can get the "conservation law" of the energy-momentum tensor density:

$$\mathfrak{T}^\mu_{\nu,\mu} - \Gamma^\sigma_{\mu\nu}\mathfrak{T}^\mu_\sigma - \tfrac{1}{2}\mathfrak{C}^\mu_{ij} R^{ij}_{\mu\nu} = 0 \tag{36}$$

where

$$R^{ij}_{\mu\nu} = \Gamma^{ij}_{\nu,\mu} - \Gamma^{ij}_{\mu,\nu} + \Gamma^i_{k\mu}\Gamma^{kj}_\nu - \Gamma^i_{k\nu}\Gamma^{kj}_\mu$$

For KGTG, $\mathfrak{C}^\mu_{ij} = \mathfrak{S}^\mu_{ij}$, the "conservation law" (36) reduces to

$$\mathfrak{T}^\mu_{\nu,\mu} - \Gamma^\sigma_{\mu\nu}\mathfrak{T}^\mu_\sigma - \tfrac{1}{2}\mathfrak{S}^\mu_{ij} R^{ij}_{\mu\nu} = 0 \tag{37}$$

and for GR, $\Gamma^{\sigma}_{\mu\nu} = \{^{\sigma}_{\mu\nu}\}$, $\mathfrak{C}^{\mu}_{ij} = 0$, the "conservation law" (36) reduces to

$$\mathfrak{T}^{\mu}_{\nu,\mu} - \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} \mathfrak{T}^{\mu}_{\sigma} = 0 \tag{38}$$

If the energy-momentum tensor density is symmetric, (36) and (37) become

$$\mathfrak{T}^{\mu}_{\nu,\mu} - \{^{\sigma}_{\mu\nu}\}\mathfrak{T}^{\mu}_{\sigma} - \tfrac{1}{2}\mathfrak{C}^{\mu}_{ij} R^{ij}_{\mu\nu} = 0 \tag{39}$$

$$\mathfrak{T}^{\mu}_{\nu,\mu} - \{^{\sigma}_{\mu\nu}\}\mathfrak{T}^{\mu}_{\sigma} - \tfrac{1}{2}\mathfrak{S}^{\mu}_{ij} R^{ij}_{\mu\nu} = 0 \tag{40}$$

respectively, since the following relations hold (Hehl *et al.*, 1976):

$$\Gamma^{\sigma}_{\mu\nu} = \{^{\sigma}_{\mu\nu}\} + K^{\sigma}_{\mu\nu}, \qquad K^{\sigma\mu}_{\nu} = -K^{\mu\sigma}_{\nu} \tag{41}$$

$$(K^{\sigma}_{\mu\nu} = T^{\sigma}_{\mu\nu} - T^{\sigma}_{\mu\nu} - T^{\sigma}_{\nu\mu})$$

3. MOMENTUM EQUATION OF MOTION AND SPIN EQUATION OF MOTION

The so-called test particle is a small body used to test the gravitational action; it is assumed that both the space extension of the test particle and its self-field of gravitation are very small and can be neglected in the discussed problem. The test particle may be a macroscopic body or a microcosmic particle; it can be represented by a certain matter field such as the Dirac field for the electron, the Maxwell field for the photon, the density field $\rho(x)$ for a macroscopic fluid, etc. Except for the scalar particles, all the other elementary particles possess intrinsic spin. Although many macroscopic bodies usually have no net intrinsic spin, their spin density is always nonzero since they are composed of elementary particles; moreover, there are other bodies having net intrinsic spin, for example, neutron stars. Considering all this, I shall regard the test particle as possessing intrinsic spin in general. For the sake of simplicity I suppose that the test particle is not acted upon by other interactions (e.g., electromagnetic force) except for the gravitational force.

If quantum effects may be neglected, the motion of a test particle in space-time can be described by a curve of four dimensions, i.e., a world line. The differential equation of a test particle's world line is called its equation of orbit. The orbital motion of a test particle is a nonquantum phenomenon in essence, so the world line is only a conditional concept for a microcosmic particle. However, the applicable range of the momentum's equation of motion is wider than that of the equation of orbit.

I use the method of Papapetrou (1951) and make the following definitions.

Momentum: $p_\nu = \int \mathfrak{T}_\nu^0 \, d^3x$.
Ordinary spin angular momentum: $S_{ij} = \int \mathfrak{S}_{ij}^0 \, d^3x$.
Generalized spin angular momentum: $C_{ij} = \int \mathfrak{C}_{ij}^0 \, d^3x$.
Additive spin angular momentum caused by Vierbein field: $b_{ij} = \int \mathfrak{B}_{ij}^0 \, d^3x$.
Additive spin angular momentum caused by frame connection field: $d_{ij} = \int \mathfrak{D}_{ij}^0 \, d^3x$.
Now, for a test particle, we can get its momentum's equation of motion

$$\frac{dp_\nu}{dt} - \Gamma^{\sigma}_{\mu\nu} p_\sigma \frac{dx^\mu}{dt} = \tfrac{1}{2} R^{ij}_{\mu\nu} C_{ij} \frac{dx^\mu}{dt} \tag{42}$$

from (36), and get its spin's equation of motion

$$\frac{dC_{ij}}{dt} = 2h^{\nu}_{[m} h_{n]\mu} p_\nu \frac{dx^\mu}{dt} + (\Gamma^{l}_{i\mu} C_{lj} + \Gamma^{l}_{j\mu} C_{il}) \frac{dx^\mu}{dt} \tag{43}$$

from (29), where t is the chosen coordinate time. Equations (42) and (43) can be written as follows:

$$\frac{dp^\nu}{d\lambda} + (\Gamma^{\nu}_{\sigma\mu} - 2T^{\nu}_{\sigma\mu}) p^\sigma \frac{dx^\mu}{d\lambda} = \tfrac{1}{2} R^{\nu}_{ij\mu} C^{ij} \frac{dx^\mu}{d\lambda} \tag{42$'$}$$

$$\frac{dC_{ij}}{d\lambda} = 2h_{[m|\nu|} h_{n]\mu} p^\nu \frac{dx^\mu}{d\lambda} + (\Gamma^{l}_{i\mu} C_{lj} + \Gamma^{l}_{j\mu} C_{il}) \frac{dx^\mu}{d\lambda} \tag{43$'$}$$

respectively, where

$$p^\nu = g^{\nu\sigma} p_\sigma, \qquad C^{ij} = \eta^{ik} \eta^{jl} C_{kl}, \qquad T^{\nu}_{\sigma\mu} = g_{\alpha\sigma} g^{\lambda\nu} T^{\alpha}_{\mu\lambda}$$

λ is a parameter along the world line of the test particle.

If the energy-momentum tensor density is symmetric, (42), (43), (42′), and (43′) become

$$\frac{dp_\nu}{dt} - \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} p_\sigma \frac{dx^\mu}{dt} = \tfrac{1}{2} R^{ij}_{\mu\nu} C_{ij} \frac{dx^\mu}{dt} \tag{44}$$

$$\frac{dC_{ij}}{dt} = (\Gamma^{l}_{i\mu} C_{lj} + \Gamma^{l}_{j\mu} C_{il}) \frac{dx^\mu}{dt} \tag{45}$$

$$\frac{dp^\nu}{d\lambda} + \left\{ \begin{matrix} \nu \\ \sigma\mu \end{matrix} \right\} p^\sigma \frac{dx^\mu}{d\lambda} = \tfrac{1}{2} R^{\nu}_{ij\mu} C^{ij} \frac{dx^\mu}{d\lambda} \tag{44$'$}$$

$$\frac{dC_{ij}}{d\lambda} = (\Gamma^{l}_{i\mu} C_{lj} + \Gamma^{l}_{j\mu} C_{il}) \frac{dx^\mu}{d\lambda} \tag{45$'$}$$

respectively.

From (30) and the definitions of C_{ij}, S_{ij}, b_{ij}, and d_{ij} we get the relation

$$C_{ij} = S_{ij} + b_{ij} + d_{ij} \tag{46}$$

This relation and the other formulas obtained in this paper mean that the value of a particle's spin would be influenced by a gravitational field in our theory. Whether this influence exists must be determined by further experimental research.

In relativistic mechanics the 4-momentum is defined by

$$p^{\nu} = m_0 c \frac{dx^{\nu}}{d\lambda}, \qquad d\lambda^2 = ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \tag{47}$$

for a massive particle and by

$$p^{\nu} = \hbar \frac{dx^{\mu}}{d\lambda} \tag{48}$$

for a massless (e.g., photon) particle (Landau and Lifshitz, 1975). Putting (47) into (42′) and (44′), we get

$$\frac{d^2x^{\nu}}{ds^2} + (\Gamma^{\nu}_{\sigma\mu} - 2T^{\nu}_{\sigma\mu}) \frac{dx^{\sigma}}{ds}\frac{dx^{\mu}}{ds} = \frac{1}{2m_0 c} R^{\nu}_{ij\mu} C^{ij} \frac{dx^{\mu}}{ds} \tag{49}$$

$$\frac{d^2x^{\nu}}{ds^2} + \left\{ \begin{matrix} \nu \\ \sigma\mu \end{matrix} \right\} \frac{dx^{\sigma}}{ds}\frac{dx^{\mu}}{ds} = \frac{1}{2m_0 c} R^{\nu}_{ij\mu} C^{ij} \frac{dx^{\mu}}{ds} \tag{50}$$

which are the equations of orbit for a massive particle according to whether the energy-momentum tensor density is asymmetric or symmetric. Putting (48) into (42′) and (44′), we get the corresponding equations of orbit for massless particle:

$$\frac{d^2x^{\nu}}{d\lambda^2} + (\Gamma^{\nu}_{\sigma\mu} - 2T^{\nu}_{\sigma\mu}) \frac{dx^{\sigma}}{d\lambda}\frac{dx^{\mu}}{d\lambda} = \frac{1}{2\hbar} R^{\nu}_{ij\mu} C^{ij} \frac{dx^{\mu}}{d\lambda} \tag{51}$$

$$\frac{d^2x^{\nu}}{d\lambda^2} + \left\{ \begin{matrix} \nu \\ \sigma\mu \end{matrix} \right\} \frac{dx^{\sigma}}{d\lambda}\frac{dx^{\mu}}{d\lambda} = \frac{1}{2\hbar} R^{\nu}_{ij\mu} C^{ij} \frac{dx^{\mu}}{d\lambda} \tag{52}$$

4. CONCLUSION

The momentum equations of motion [(42) or (42′), (44) or (44′)] and the spin equations of motion [(43) or (43′), (45) or (45′)] for test particles possessing different spin in space-time with torsion have been derived from the most general functional form of $\mathscr{L}_M$ given by (3). The application ranges of these equations are very wide; I indicate the following special cases.

4.1. The Case of KGTG

In this case, $b_{ij}=0$, $d_{ij}=0$, $c_{ij}=s_{ij}$, then, c_{ij} in (42)-(45) [or (42′)-(45′)] will be reduced to s_{ij}. If the test particle is a scalar particle, $s_{ij}=0$.

4.2. The Case of GR

In this case, $c_{ij}=0$, $d_{ij}=0$, and

$$s_{ij}=-b_{ij}\neq 0 \qquad \text{for particle with spin}$$

$$s_{ij}=b_{ij}=0 \qquad \text{for scalar particle}$$

Then, c_{ij} in (44), (45) [or (44′), (45′)] will be reduced to zero. Incidentally, the energy-momentum tensor density in GR is always symmetric, so (42), (43) [or (42′), (43′)] are identical with (44), (45) [or (44′), (45′)] respectively.

4.3. Case of the Photon and the Quantum of the Yang–Mills Field in Space-Time with Torsion but without the Action of Torsion

The photon and the quantum of the Yang-Mills field are vector particles. If these particles do not act directly by torsion, then their Lagrangian density in space-time with torsion must be also represented by

$$\mathscr{L}_M(x)=\mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), h^i_{\mu,\nu}(x), \eta_{ij}(x)]$$

in order to be compatible both with the local gauge invariance and minimal coupling (Yasskin and Stoeger, 1980; Chen and Jiang, 1989). Therefore, in this case, the equations of motion for the photon and the quantum of the Yang-Mills field are the same as in GR.

4.4. The Case with $\mathscr{L}_M(x)=\mathscr{L}_M[\psi(x), \psi_{,\mu}(x), h^i_\mu(x), h^i_{\mu,\nu}(x), \Gamma^{ij}_\mu(x), \eta_{ij}(x), \gamma^i(x)]$

In this case $d_{ij}=0$ and $c_{ij}=s_{ij}+b_{ij}$. As an example of this case, the equations of motion for the photon and for scalar particles are discussed in detail elsewhere (Chen and Jiang, 1989).

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